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ABSTRACT : While the description and use of incremental methods in finite elasticity have received considerable attention in the engineering literature, it seems that its numerical analysis is an open problem. In this paper, we prove the convergence of such methods when applied to a class of problems in nonlinear, three dimensional, elasticity.

RESUME : Alors que la description et l'utilisation de méthodes incrémentales en élasticité finie sont très répandues parmi les ouvrages à destination des ingénieurs, il semble que l'analyse numérique de ces méthodes est un problème ouvert. Dans ce travail, nous démontrons la convergence de ces méthodes lorsqu'elles sont appliquées à une classe de problèmes d'élasticité non linéaire tridimensionnelle.

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I - THE BOUNDARY-VALUE PROBLEM OF NONLINEAR, THREE-DIMENSIONAL ELASTICITY

Consider an isotropic, homogeneous, elastic body subjected to body forces in its interior, and to surface forces on a portion of its boundary, the displacement being imposed on the remaining portion of the boundary. A mathematical model for finding the equilibrium of such a mechanical system is a nonlinear, boundary-value problem which we now briefly describe in a formal way (for details, see for instance CIARLET [1983], GURTIN [1981], MARSDEN & HUGHES [1982], WANG & TRUESDELL [1973], WASHIZU [1975]).

Let Ω be a bounded open and connected subset of the space $\mathbb{R}^3$, whose boundary Γ is assumed to be sufficiently smooth for all subsequent purposes. We denote by $\underset{\sim}{\nu} = (\nu_i)$ the unit outer normal vector along Γ.

We consider a body which occupies the set $\bar{\Omega}$ in the absence of applied forces, so that $\bar{\Omega}$ is called the *reference configuration*. The body is subjected to body forces of *dead loading type*, represented by a vector field $\underset{\sim}{f} = (f_i)$ given over Ω (as a rule, Latin indices : $i,j,k,p,\ldots$, take their values in the set $\{1,2,3\}$) and to surface forces, again of *dead loading* type, represented by a vector field $\underset{\sim}{g} = (g_i)$ given over a portion Γ_1 of Γ. The fields $\underset{\sim}{f}$ and $\underset{\sim}{g}$ respectively measure the density per unit volume (in Ω) and the density per unit area (along Γ_1) of both kinds of applied forces. The problem then consists in solving the following nonlinear boundary value problem for the *displacement field* $\underset{\sim}{u} = (u_i)$ and the (second) *Piola-Kirchhoff stress tensor field* $\underset{\sim}{\sigma} = (\sigma_{ij})$ (in what follows, the repeated index convention for summation is systematically used in conjunction with the above rule for Latin indices and the notation ∂_j stands for the usual partial derivative $\frac{\partial}{\partial x_j}$; the tensor $\underset{\sim}{e}(\underset{\sim}{u})$ is defined in (1.5) below) :

$$(1.1) \qquad -\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \text{ in } \Omega ,$$

$$(1.2) \qquad \sigma_{ij} = \tilde{\sigma}_{ij}(\underset{\sim}{e}(\underset{\sim}{u})) \text{ in } \Omega ,$$

$$(1.3) \qquad (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\nu_j = g_i \text{ on } \Gamma_1 ,$$

$$(1.4) \qquad \underset{\sim}{u} = \underset{\sim}{o} \text{ on } \Gamma_o .$$

Equations (1.1) and (1.3) represent the *equilibrium equations*, expressed in terms of the stress tensor $\underset{\sim}{\sigma}$, in the reference configuration. Equations (1.2) represent the *constitutive equation* of an *elastic, homogeneous, isotropic*, material : At each point of the reference configuration, the stress tensor

$\underset{\sim}{\sigma} = (\sigma_{ij})$ is a *known* function

$$\tilde{\underset{\sim}{\sigma}}(\underset{\sim}{e}(\underset{\sim}{u})) = (\tilde{\sigma}_{ij}(\underset{\sim}{e}(\underset{\sim}{u})))$$

of the *strain tensor*

$$(1.5) \qquad \underset{\sim}{e}(\underset{\sim}{u}) = (e_{ij}(\underset{\sim}{u})), \text{ with } 2e_{ij}(\underset{\sim}{u}) = \partial_i u_j + \partial_j u_i + \partial_i u_m \partial_j u_m ,$$

at the same point. According to the Rivlin-Ericksen theorem (cf.e.g. WANG & TRUESDELL [1973]), the function $\tilde{\underset{\sim}{\sigma}}$ is of the form

$$(1.6) \qquad \tilde{\underset{\sim}{\sigma}}(\underset{\sim}{e}) = \gamma_o(\underset{\sim}{C})\underset{\sim}{I} + \gamma_1(\underset{\sim}{C})\underset{\sim}{C} + \gamma_2(\underset{\sim}{C})\underset{\sim}{C}^2 ,$$

where the matrix C is defined by the relation

$$(1.7) \qquad \underset{\sim}{C} = \underset{\sim}{I} + 2\underset{\sim}{e} ,$$

and the functions $\gamma_\alpha(\underset{\sim}{C})$ are symmetric functions of the principal invariants of the matrix $\underset{\sim}{C}$. If we make the assumption that *the reference configuration is a natural state*, i.e.,

$$(1.8) \qquad \tilde{\underset{\sim}{\sigma}}(\underset{\sim}{0}) = \underset{\sim}{0} ,$$

then it is easily derived from (1.6) and (1.8) that, if the functions $\gamma_\alpha(\underset{\sim}{C})$ are differentiable at the point $\underset{\sim}{C} = \underset{\sim}{I}$,

$$(1.9) \qquad \underset{\sim}{\sigma} = \tilde{\underset{\sim}{\sigma}}(\underset{\sim}{e}) = \lambda(\text{tr } \underset{\sim}{e})\underset{\sim}{I} + 2\mu\, \underset{\sim}{e} + 0(|\underset{\sim}{e}|) ,$$

where $|\cdot|$ denotes any norm in $\mathbb{R}^9$, and λ et μ are two constants, known as the *Lamé coefficients* of the constituting material of the body. These constants will be assumed here to satisfy the inequalities

$$(1.10) \qquad \lambda \geq 0 \quad , \quad \mu > 0 ,$$

as is indeed the case for most "real-life" elastic, homogeneous, isotropic, materials.

A constitutive equation commonly used in practical applications corresponds to the deletion of the higher-order terms in (1.9), i.e., it reduces to

$$(1.11) \qquad \underset{\sim}{\sigma} = \lambda(\text{tr } \underset{\sim}{e})\underset{\sim}{I} + 2\mu\, \underset{\sim}{e} ,$$

or equivalently, componentwise :

(1.12) $\sigma_{ij} = a_{ijpq} e_{pq}$, with $a_{ijpq} \stackrel{\text{def}}{=} \lambda\delta_{ij}\delta_{pq} + 2\mu\delta_{ip}\delta_{jq}$.

We shall refer to a material obeying such a constitutive equation as a *St Venant-Kirchhoff* material. Let us now briefly discuss an existence result for the boundary value problem (1.1)-(1.4), which relies on the *implicit function theorem*. Another successful approach consists in minimizing the associated energy (in this direction, see the famous paper of BALL [1977]), but this second approach does not seem to be appropriate for our present purposes. For a more detailed discussion, see CIARLET [1982].

To fix ideas, consider the case of a St Venant-Kirchhoff material (this is not a restriction) and of the "*pure displacement problem*" (this is an *essential* restriction) that is, $\Gamma = \Gamma_o$. Then the boundary value problem (1.1)-(1.4) reduces in this case to

(1.13) $- \partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i$ in Ω ,

(1.14) $\sigma_{ij} = a_{ijpq} e_{pq}(\underset{\sim}{u})$ in Ω ,

(1.15) $\underset{\sim}{u} = \underset{\sim}{o}$ on Γ ,

or, in terms of the displacement vector only (taking into account the symmetry property $a_{ijkl} = a_{ijlk}$) :

(1.16) $$\mathcal{B}_i(\underset{\sim}{u}) \stackrel{\text{def}}{=} - \partial_j(a_{ijpq}\partial_p u_q + \frac{1}{2} a_{ijpq}\partial_p u_m \partial_q u_m + a_{kjpq}\partial_p u_q \partial_k u_i + \frac{1}{2} a_{kjpq}\partial_p u_m \partial_q u_m \partial_k u_i) = f_i \text{ in } \Omega ,$$

(1.17) $u_i = 0$ on Γ .

Since the Sobolev space $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is an algebra for $p > 3$ (see e.g. ADAMS [1975, p. 115]), the nonlinear mapping $\underset{\sim}{\mathcal{B}}$ defined in (1.16) maps the space $\underset{\sim}{W}^{2,p}(\Omega) \stackrel{\text{def}}{=} (W^{2,p}(\Omega))^3$ into the space $\underset{\sim}{L}^p(\Omega) \stackrel{\text{def}}{=} (L^p(\Omega))^3$, and besides it is (infinitely) differentiable between these two spaces, as a sum of continuous multilinear mappings. Equations (1.16)-(1.17) then consist in finding a function

(1.18) $\underset{\sim}{u} \in \underset{\sim}{V}^p(\Omega) \stackrel{\text{def}}{=} \{\underset{\sim}{v} = \underset{\sim}{W}^{2,p}(\Omega) ; \underset{\sim}{v} = \underset{\sim}{O} \text{ on } \Gamma\}$

such that

$$(1.19)\quad \underset{\sim}{\mathcal{B}}(\underset{\sim}{u}) = \underset{\sim}{f} \ ,$$

where the mapping

$$(1.20)\quad \underset{\sim}{\mathcal{B}} = (\mathcal{B}_i) : \underset{\sim}{V}^p(\Omega) \to \underset{\sim}{L}^p(\Omega)$$

is defined as in (1.16). In order to use the *implicit function theorem* in a neighborhood of the origin in both spaces $\underset{\sim}{V}^p(\Omega)$ and $\underset{\sim}{L}^p(\Omega)$ (since $\underset{\sim}{u} = \underset{\sim}{0}$ is clearly a solution corresponding to $\underset{\sim}{f} = \underset{\sim}{0}$), we must verify that the derivative $\underset{\sim}{\mathcal{B}}'(\underset{\sim}{0})$ is an isomorphism between the spaces $\underset{\sim}{V}^p(\Omega)$ and $\underset{\sim}{L}^p(\Omega)$. But the equation

$$(1.21)\quad \underset{\sim}{\mathcal{B}}'(\underset{\sim}{0})\underset{\sim}{u} = \underset{\sim}{f}$$

is precisely the *system of linear elasticity* :

$$(1.22)\quad -\partial_j(a_{ijpq}\varepsilon_{pq}(\underset{\sim}{u})) = f_i \text{ in } \Omega \ ,$$

$$(1.23)\quad \underset{\sim}{u} = \underset{\sim}{0} \text{ on } \Gamma \ ,$$

where the tensor $\underset{\sim}{\varepsilon}(\underset{\sim}{u}) = (\varepsilon_{ij}(\underset{\sim}{u}))$, with

$$(1.24)\quad 2\varepsilon_{ij}(\underset{\sim}{u}) \overset{\text{def}}{=} \partial_i u_j + \partial_j u_i \ ,$$

is the so-called *linearized strain tensor*. Since the operator $\underset{\sim}{\mathcal{B}}'(\underset{\sim}{0}) : \underset{\sim}{V}^p(\Omega) \to \underset{\sim}{L}^p(\Omega)$ is clearly continuous and one-to-one, it remains to verify that it is *onto*, i.e., we need a *regularity result* of the form :

$$(1.25)\quad \underset{\sim}{\mathcal{B}}'(\underset{\sim}{0})\underset{\sim}{u} \in \underset{\sim}{L}^p(\Omega) \Longrightarrow \underset{\sim}{u} \in \underset{\sim}{V}^p(\Omega) \ ,$$

where $\underset{\sim}{u}$ is the solution (known to exist by the variational theory in the space $(H_0^1(\Omega))^3$) of (1.22)-(1.23). As shown by NEČAS [1967] for $p = 2$ and by GEYMONAT [1965] for $1 < p < +\infty$, the regularity result (1.25) holds, basically because the boundary condition *does not change along* Γ . *It is the lack of such a regularity result in the case of the genuine mixed displacement-problem* (1.1)-(1.4) *that prevents the use of the implicit function theorem in the general case* .

This type of analysis has been applied by CIARLET & DESTUYNDER [1979] to the St Venant-Kirchhoff material along the lines indicated here ; it has also been independently applied to more general constitutive equation by MARSDEN & HUGHES [1978] and VALENT [1979], who proved the following existence result :

THEOREM 1 : *Assume that the functions* $\tilde{\sigma}_{ij}$ *which appear in the constitutive equation* (1.28) *are smooth enough at the point* $\underset{\sim}{e} = \underset{\sim}{0}$ *and that there exists a constant* μ *such that*

$$(1.26) \qquad \mu > 0 \text{ and } \frac{\partial \tilde{\sigma}_{ij}}{\partial e_{kl}}(\underset{\sim}{0})\xi_{ij}\xi_{kl} \geq 2\mu\xi_{ij}\xi_{ij} \quad ,$$

for all symmetric tensors $\underset{\sim}{\xi} = (\xi_{ij})$. *Then for each number* $p > 3$, *there exist a neighborhood* $\underset{\sim}{\mathfrak{F}}^p$ *of* $\underset{\sim}{0}$ *in* $\underset{\sim}{L}^p(\Omega)$ *and a neighborhood* $\underset{\sim}{\mathcal{U}}^p$ *of* $\underset{\sim}{0}$ *in* $\underset{\sim}{V}^p(\Omega)$ *such that, for each* $\underset{\sim}{f} \in \underset{\sim}{\mathfrak{F}}^p$, *the boundary value problem*

$$(1.27) \qquad -\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \quad \text{in } \Omega ,$$

$$(1.28) \qquad \sigma_{ij} = \tilde{\sigma}_{ij}(\underset{\sim}{e}(\underset{\sim}{u})) \text{ in } \Omega ,$$

$$(1.29) \qquad \underset{\sim}{u} = \underset{\sim}{0} \text{ on } \Gamma ,$$

has exactly one solution $\underset{\sim}{u}(\underset{\sim}{f})$ *in* $\underset{\sim}{\mathcal{U}}^p$. □

2 - DESCRIPTION OF THE INCREMENTAL METHOD

Let us now review the application of the *incremental method* to the nonlinear boundary value problem described in Sect.1, as it is commonly presented in the engineering or mechanical litterature. We describe here the so-called *total Lagrangian method* (as it is presented for instance in WASHIZU [1975, Appendix I, Section 9]), in the special case of a St Venant-Kirchhoff material. For similar or related discussions, see ARGYRIS & KLEIBER [1977], MASON [1980], PIAN [1976].

Our problem takes the form of eqs. (1.13)-(1.15), which we rewrite here for convenience :

$$(2.1) \qquad -\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \text{ in } \Omega ,$$

$$(2.2) \qquad \sigma_{ij} = a_{ijpq}e_{pq}(\underset{\sim}{u}) \text{ in } \Omega ,$$

$$(2.3) \qquad \underset{\sim}{u} = \underset{\sim}{0} \text{ on } \Gamma ,$$

with

(2.4) $a_{ijpq} = \lambda\delta_{ij}\delta_{pq} + 2\mu\delta_{ip}\delta_{jq}$, $\lambda \geq 0$, $\mu > 0$.

We also assume that there exists a number $p > 3$ such that the whole segment $[\underset{\sim}{0},\underset{\sim}{f}]$ belongs to the neighborhood $\mathfrak{F}^p$ given in Theorem 1.

The basic idea of the *incremental method* is to let the body forces vary by "small" *increments*

(2.5) $\Delta\underset{\sim}{f} = \frac{1}{N}\underset{\sim}{f}$, N : a given "large" integer ,

from $\underset{\sim}{0}$ to the given force $\underset{\sim}{f}$, and to recursively compute *approximations* $\underset{\sim}{u}^n$ to the *exact solutions*

(2.6) $\underset{\sim}{U}^n \overset{\text{def}}{=} \underset{\sim}{u}(\underset{\sim}{f}^n)$

corresponding to the successive forces

(2.7) $\underset{\sim}{f}^n = n\Delta\underset{\sim}{f}$, $1 \leq n \leq N$

(such solutions $\underset{\sim}{u}(\underset{\sim}{f}^n)$ exist by Theorem 1), each approximation being computed by an appropriate "linearization around the previous approximation". In this fashion the nonlinear problem is approximated by a sequence of linear problems.

Using definitions (2.6), let

(2.8) $\Delta U_i^n \overset{\text{def}}{=} U_i^{n+1} - U_i^n$,

(2.9) $\Delta e_{ij}^n \overset{\text{def}}{=} e_{ij}(U^{n+1}) - e_{ij}(U^n)$,

denote the *displacement increments* and the corresponding *strain increments*, respectively, for $0 \leq n \leq N-1$, so that the corresponding *stress increments* take the form

(2.10) $\Sigma_{ij}^{n+1} - \Sigma_{ij}^n = a_{ijpq}\,\Delta e_{pq}^n$,

with

(2.11) $\Sigma_{ij}^n \overset{\text{def}}{=} a_{ijpq}e_{pq}(U^n)$.

Since, by definition,

$$(2.12) \qquad - \partial_j(\Sigma_{ij}^{n+1} + \Sigma_{kj}^{n+1}\, \partial_k U_i^{n+1}) = f_i^{n+1} = f_i^n + \Delta f_i \quad \text{in } \Omega \quad ,$$

$$(2.13) \qquad - \partial_j(\Sigma_{ij}^{n} + \Sigma_{kj}^{n}\, \partial_k U_i^{n}) = f_i^{n} \text{ in } \Omega \ ,$$

where Σ_{ij}^{n+1} and Σ_{ij}^{n} are related through relation (2.10), one obtains, after subtracting eqs. (2.13) from eqs. (2.12), that *the* n-th *displacement increment* $\Delta \underset{\sim}{U}^n = (\Delta U_i^n)$ *satisfies the following boundary value problem* (notice that, up to this point, *no approximation has been made)* :

$$(2.14) \qquad - \partial_j(a_{ijpq}\Delta e_{pq}^n + \Sigma_{kj}^n \partial_k \Delta U_i^n + a_{kjpq}\Delta e_{pq}^n \partial_k U_i^n + a_{kjpq}\Delta e_{pq}^n \partial_k \Delta U_i^n) = \Delta f_i \text{ in } \Omega,$$

$$(2.15) \qquad 2\Delta e_{pq}^n = \partial_p \Delta U_q^n + \partial_q \Delta U_p^n + \partial_p \Delta U_m^n \partial_q U_m^n + \partial_p U_m^n \partial_q \Delta U_m^n + \partial_p \Delta U_m^n \partial_q \Delta U_m^n \quad \text{in} \quad \Omega,$$

$$(2.16) \qquad \Sigma_{kj}^n = a_{kjpq} e_{pq}(U^n) \text{ in } \Omega \ ,$$

$$(2.17) \qquad \Delta \underset{\sim}{U}^n = \underset{\sim}{0} \text{ on } \Gamma \ .$$

We are now in a position to define an *approximate problem* : Considering that the n-th displacement $\underset{\sim}{U}^n$ is known, and using eqs. (2.15)-(2.16) in eqs. (2.14), we obtain a nonlinear boundary value problem with respect to the unknown vector $\Delta \underset{\sim}{U}^n$. Then the approximation simply consists in *deleting all the terms which are nonlinear with respect to the unknown* $\Delta \underset{\sim}{U}^n$ *in the resulting problem.* In this fashion, we obtain that the n-th *approximate displacement increment* $\delta \underset{\sim}{u}^n$ should be solution of the following *linear* boundary value problem, where $\underset{\sim}{u}^n$ likewise denotes the n-th *approximate displacement* vector :

$$(2.18) \qquad - \partial_j(a_{ijpq}\partial_p \delta u_q^n + a_{ijpq}\partial_p u_m^n \partial_q \delta u_m^n + a_{kjpq}\partial_k u_i^n \partial_p \delta u_q^n +$$
$$+ a_{kjpq}\partial_k u_i^n \partial_p u_m^n \partial_q \delta u_m^n + a_{kjpq} e_{pq}(\underset{\sim}{u}^n)\partial_k \delta u_i^n) = \Delta f_i \text{ in } \Omega \ ,$$

$$(2.19) \qquad \delta \underset{\sim}{u}^n = \underset{\sim}{0} \quad \text{on } \Gamma \ .$$

This is exactly the problem obtained, in an *equivalent* variational form, in WASHIZU [1975, eq. (I-9.42), p. 393].

Provided the boundary value problem (2.18)-(2.19) has a unique solution (this will be proved in Theorem 3 below), we define the (n+1)-st approximate displacement

$$(2.20) \qquad \underset{\sim}{u}^{n+1} = \underset{\sim}{u}^n + \delta \underset{\sim}{u}^n \quad ,$$

which in turn allows us to similarly compute the (n+1)-st approximate displacement increment, etc. In this fashion the *incremental method* is completely defined ; it ends with the computation of the N-th approximate displacement $\underset{\sim}{u}^N$, which will be proved later to approach the exact solution $\underset{\sim}{u}(\underset{\sim}{f})$ as the integer N approaches infinity.

3 - CONVERGENCE OF THE INCREMENTAL METHOD

Let us first prove a number of properties (useful for the sequel) of the nonlinear mapping

$$(3.1) \qquad \underset{\sim}{\mathcal{B}} : \underset{\sim}{V}^p(\Omega) \overset{\text{def}}{=} \{\underset{\sim}{v} \in \underset{\sim}{W}^{2,p}(\Omega) \ ; \ \underset{\sim}{v} = \underset{\sim}{0} \text{ on } \Gamma\} \to \underset{\sim}{L}^p(\Omega)$$

defined (cf. (1.16)) by

$$(3.2) \qquad \mathcal{B}_i(\underset{\sim}{u}) = -\partial_j(a_{ijpq}\partial_p u_q + \frac{1}{2} a_{ijpq}\partial_p u_m \partial_q u_m + a_{kjpq}\partial_p u_q \partial_k u_i + \frac{1}{2} a_{kjpq}\partial_p u_m \partial_q u_m \partial_k u_i) ,$$

with

$$(3.3) \qquad a_{ijpq} = \lambda\delta_{ij}\delta_{pq} + 2\mu\delta_{ip}\delta_{jq} , \quad \lambda \geq 0 , \quad \mu > 0 .$$

As already observed, because the space $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is an algebra for each number $p > 3$, the mapping $\underset{\sim}{\mathcal{B}} : \underset{\sim}{V}^p(\Omega) \to \underset{\sim}{L}^p(\Omega)$ is well-defined and of class $\mathcal{C}^\infty$ for such values of p.

For each integer $m \geq 0$ and each number $p \geq 1$, we denote by

$$\|v\|_{m,p,\Omega} = \{ \int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha v|^p dx\}^{1/p}$$

the norm of the Sobolev space $W^{m,p}(\Omega)$. If X and Y are two normed vector spaces, we let

$$\text{Isom}(X;Y) = \{A \in \mathcal{L}(X;Y) \ ; \ A^{-1} \text{ exists and } A^{-1} \in \mathcal{L}(Y;X)\} ,$$

and for notational brevity we denote by the same symbol $\|\cdot\|$ the norms of the spaces $\mathcal{L}(X;Y)$, $\mathcal{L}(Y;X)$, $\mathcal{L}_2(X;Y)$, etc... (such as in (3.6)-(3.7) below). Finally, we let $\mathcal{B}'(u) \in \mathcal{L}(X;Y)$, $\mathcal{B}''(u) \in \mathcal{L}_2(X;Y)$, etc... denote the successive Fréchet derivatives at a point $u \in X$ (when they exist) of a mapping $\mathcal{B} : X \to Y$.

LEMMA 1. *Let a number* $p > 3$ *be fixed. Then there exists a number* $\rho_o(p) > 0$ *such that, for any number* ρ *such that*

$$(3.4) \qquad \rho < \rho_o(p)\ ,$$

one has

$$(3.5) \qquad \underset{\sim}{v} \in \underset{\sim}{B}^p_\rho \Rightarrow \underset{\sim}{B}'(\underset{\sim}{v}) \in \text{Isom}\ (\underset{\sim}{V}^p(\Omega)\ ;\ \underset{\sim}{L}^p(\Omega))\ ,$$

$$(3.6) \qquad \gamma_\rho \overset{\text{def}}{=} \sup_{\underset{\sim}{v}\in \underset{\sim}{B}^p_\rho} \|\{\underset{\sim}{B}'(\underset{\sim}{v})\}^{-1}\| < +\infty\ ,$$

$$(3.7) \qquad L_\rho \overset{\text{def}}{=} \sup_{\substack{\underset{\sim}{v},\underset{\sim}{w} \in \underset{\sim}{B}^p_\rho \\ \underset{\sim}{v} \neq \underset{\sim}{w}}} \frac{\|\{\underset{\sim}{B}'(\underset{\sim}{v})\}^{-1} - \{\underset{\sim}{B}'(\underset{\sim}{w})\}^{-1}\|}{\|\underset{\sim}{v} - \underset{\sim}{w}\|_{2,p,\Omega}} < +\infty\ ,$$

where

$$(3.8) \qquad \underset{\sim}{B}^p_\rho \overset{\text{def}}{=} \{\underset{\sim}{v} \in \underset{\sim}{V}^p(\Omega)\ ;\ \|\underset{\sim}{v}\|_{2,p,\Omega} \le \rho\}.$$

Proof : Using the fact that the mapping $\underset{\sim}{B}$ is a sum of continuous linear, bilinear, and trilinear mappings, we exactly have, for arbitrary functions $\underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{V}^p(\Omega)$,

$$\underset{\sim}{B}(\underset{\sim}{u}+\underset{\sim}{v}) = \underset{\sim}{B}(\underset{\sim}{u}) + \underset{\sim}{B}'(\underset{\sim}{u})\underset{\sim}{v} + \frac{1}{2}\underset{\sim}{B}''(\underset{\sim}{u})(\underset{\sim}{v},\underset{\sim}{v}) + \frac{1}{6}\underset{\sim}{B}'''(\underset{\sim}{u})(\underset{\sim}{v},\underset{\sim}{v},\underset{\sim}{v})\ .$$

Expanding $\underset{\sim}{B}(\underset{\sim}{u}+\underset{\sim}{v})$ with the help of (3.2) and using the symmetry of the second derivative $\underset{\sim}{B}''(\underset{\sim}{u})$, we find that for arbitrary functions $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w} \in \underset{\sim}{V}^p(\Omega)$, each function $B''_i(\underset{\sim}{u})(\underset{\sim}{v},\underset{\sim}{w}) \in L^p(\Omega)$ is a sum of terms of either form (up to multiplicative constants involving the Lamé constants λ,μ of (3.3)) :

$$\partial_j(\partial_p v_m \partial_q w_m) \text{ or } \partial_j(\partial_l u_i \partial_p v_m \partial_q w_m)\ ,$$

so that, using again the algebricity of the Sobolev space $W^{1,p}(\Omega)$ for $p > 3$, we find that there exists a constant C such that

$$\|\underset{\sim}{B}''(\underset{\sim}{u})(\underset{\sim}{v},\underset{\sim}{w})\|_{o,p,\Omega} \le C(1 + \|\underset{\sim}{u}\|_{2,p,\Omega})\|\underset{\sim}{v}\|_{2,p,\Omega}\|\underset{\sim}{w}\|_{2,p,\Omega}$$

for all $\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w} \in \underset{\sim}{W}^{2,p}(\Omega)$. Therefore,

$$\|\underset{\sim}{B}''(\underset{\sim}{u})\| = \sup_{\substack{\underset{\sim}{v}\neq \underset{\sim}{0} \\ \underset{\sim}{w}\neq \underset{\sim}{0}}} \frac{\|\underset{\sim}{B}''(\underset{\sim}{u})(\underset{\sim}{v},\underset{\sim}{w})\|_{o,p,\Omega}}{\|\underset{\sim}{v}\|_{2,p,\Omega}\|\underset{\sim}{w}\|_{2,p,\Omega}} \le C(1 + \|\underset{\sim}{u}\|_{2,p,\Omega})$$

and the function

$$M : \rho \geq 0 \rightarrow M(\rho) \stackrel{\text{def}}{=} \sup_{\|\underset{\sim}{u}\|_{2,p,\Omega} \leq \rho} \|\underset{\sim}{B}''(\underset{\sim}{u})\|$$

is thus well-defined and non-decreasing on $[0,+\infty)$. Besides,

$$\|\underset{\sim}{v}\|_{2,p,\Omega} \leq \rho \Rightarrow \|\underset{\sim}{B}'(\underset{\sim}{v}) - \underset{\sim}{B}'(\underset{\sim}{0})\| \leq \rho M(\rho) \quad ,$$

by the mean-value theorem. Since (cf. Sect. 1)

$$\underset{\sim}{B}'(\underset{\sim}{0}) \in \mathrm{Isom}(\underset{\sim}{V}^p(\Omega) \; ; \; \underset{\sim}{L}^p(\Omega)) \; ,$$

we deduce from the identity

$$\underset{\sim}{B}'(\underset{\sim}{v}) = \underset{\sim}{B}'(\underset{\sim}{0}) \{\underset{\sim}{I} + \{\underset{\sim}{B}'(\underset{\sim}{0})\}^{-1}(\underset{\sim}{B}'(\underset{\sim}{v}) - \underset{\sim}{B}'(\underset{\sim}{0}))\} \; ,$$

that, if (cf. (3.4))

$$\|\underset{\sim}{v}\|_{2,p,\Omega} \leq \rho \text{ with } \rho M(\rho) < \gamma_o^{-1} \text{ and } \gamma_o = \|\{\underset{\sim}{B}'(\underset{\sim}{0})\}^{-1}\| \; ,$$

then the mapping $\underset{\sim}{B}'(\underset{\sim}{v}) : \underset{\sim}{V}^p(\Omega) \rightarrow \underset{\sim}{L}^p(\Omega)$ is an isomorphism, and

$$\|\{\underset{\sim}{B}'(\underset{\sim}{v})\}^{-1}\| \leq \frac{\gamma_o}{1-\gamma_o \rho M(\rho)} \quad ,$$

so that relations (3.5) and (3.6) are proved. If $\underset{\sim}{v}, \underset{\sim}{w} \in \underset{\sim}{B}_\rho^p$, we can write the equality

$$\{\underset{\sim}{B}'(\underset{\sim}{v})\}^{-1} - \{\underset{\sim}{B}'(\underset{\sim}{w})\}^{-1} = \{\underset{\sim}{B}'(\underset{\sim}{v})\}^{-1}(\underset{\sim}{B}'(\underset{\sim}{w}) - \underset{\sim}{B}'(\underset{\sim}{v}))\{\underset{\sim}{B}'(\underset{\sim}{w})\}^{-1}$$

which, by another application of the mean value theorem, proves relation (3.7) with the upper bound $2\rho M(\rho)\gamma_\rho^2$ for the number L_ρ. □

As our first step towards the analysis of the incremental method, we establish a connection between the boundary value problem under consideration and the solution of an appropriate *differential equation* in the space $\underset{\sim}{V}^p(\Omega)$. In what follows, the notations $\rho_o(p)$, γ_p, $\underset{\sim}{B}_\rho^p$ have the same meaning as in Lemma 1.

<u>THEOREM 2</u> . *Let a number* $p > 3$ *be fixed, and let* ρ *be any number which satisfies*

(3.9) $\qquad 0 < \rho < \rho_o(p)$

Then, if

(3.10) $\quad \|\underset{\sim}{f}\|_{o,p,\Omega} \leq \rho\gamma_\rho^{-1}$,

the differential equation : Find

(3.11) $\quad \tilde{\underset{\sim}{u}} : \lambda \in [0,1] \to \tilde{\underset{\sim}{u}}(\lambda) \in \underset{\sim}{B}_\rho^p$

such that

(3.12) $\quad \tilde{\underset{\sim}{u}}'(\lambda) = \{\underset{\sim}{B}'(\tilde{\underset{\sim}{u}}(\lambda))\}^{-1}\underset{\sim}{f} \quad , \quad 0 \leq \lambda \leq 1$,

(3.13) $\quad \tilde{\underset{\sim}{u}}(0) = \underset{\sim}{0}$

has one and only solution, which in addition satisfies

(3.14) $\quad \underset{\sim}{B}(\tilde{\underset{\sim}{u}}(\lambda)) = \lambda\underset{\sim}{f} \quad , \quad 0 \leq \lambda \leq 1$.

Proof : We shall simply give a very brief outline of the proof, which is well-known (see e.g. CROUZEIX-MIGNOT [1983]).

Relation (3.5) of Lemma 1 insures that the mapping

$$\Phi : \tilde{\underset{\sim}{v}} \in \mathcal{C}^o([0,1]; \underset{\sim}{B}_\rho^p) \to \mathcal{C}^o([0,1]; \underset{\sim}{V}^p(\Omega))$$

with

$$\Phi(\tilde{\underset{\sim}{v}})(\lambda) = \int_o^\lambda \{\underset{\sim}{B}'(\tilde{\underset{\sim}{v}}(\mu))\}^{-1}\underset{\sim}{f}\, d\mu \ , \quad 0 \leq \lambda \leq 1$$

is well-defined, and condition (3.10) together with relation (3.6) of Lemma 1 shows that Φ maps the complete metric space

$$\mathcal{X} \overset{\text{def}}{=} \mathcal{C}^o([0,1]; B_\rho^p)$$

into *itself*. Using next relation (3.7) of Lemma 1, one

shows that some iterate Φ^k , $k \geq 1$, of the mapping Φ is a contraction of the space $\mathcal{X}$, so that Φ has one, and only one, fixed point in the space $\mathcal{X}$, which is a solution of the differential equation (3.11)-(3.13). Conversely, any solution of the differential equation is clearly a fixed point of the mapping Φ in the space $\mathcal{X}$.

To show that such a solution $\tilde{u}$ satisfies $\mathcal{B}(\tilde{u}(\lambda)) = \lambda f$, $0 \leq \lambda \leq 1$, observe that, by (3.12),

$$0 = \mathcal{B}'(\tilde{u}(\lambda))\tilde{u}'(\lambda) - f = \frac{d}{d\lambda}\{\mathcal{B}(u(\lambda)) - \lambda f\} , \quad 0 \leq \lambda \leq 1 .$$

This shows that the mapping $\lambda \to (\mathcal{B}(u(\lambda)) - \lambda f) \in L^p(\Omega)$ is constant on the interval $]0,1[$ and thus on the closed interval by continuity ; finally, condition (3.13) shows that this constant is 0 , and the proof is complete.

□

The next, and final, step consists in establishing that the *incremental method* described in Sect. 2 is nothing but *Euler's method* in disguise, for the discretization of the differential equation. (3.11)-(3.13), thereby providing a proof of the convergence of the method. Again the notations $\rho_o(p)$, γ_ρ, B^p_ρ have the same meaning as in Lemma 1.

<u>THEOREM 3</u> . *Let a number* $p > 3$ *be fixed and let* ρ *be any number satisfying*

(3.15) $\quad 0 < \rho < \rho_o(p)$.

Then, if

(3.16) $\quad \|f\|_{o,p,\Omega} \leq \rho\gamma_\rho^{-1}$,

Euler's method : Find a finite sequence

(3.17) $\quad u^n \in B^p_\rho$, $0 \leq n \leq N$, N *: a given integer,*

such that

(3.18) $\quad N(u^{n+1} - u^n) = \{\mathcal{B}'(u^n)\}^{-1} f$, $\quad 0 \leq n \leq N-1$,

(3.19) $\quad u^o = 0$,

is well-defined, and besides the sequence $(u^n)_{n=0}^N$ *coincides with the sequence constructed through eqs.* (2.18)-(2.20). *Finally, there exists a constant* $\mathcal{C}_\rho$ *such that*

(3.20) $\quad \max\limits_{0 \leq n \leq N} \|U^n - u^n\|_{2,p,\Omega} = \dfrac{\mathcal{C}_\rho}{N}$

where

$$(3.21) \qquad \underset{\sim}{U}^n = \tilde{\underset{\sim}{u}}\left(\frac{n}{N}\right), \quad 0 \le n \le N ,$$

denotes for each integer $n = 0,1,\dots,N$, *the unique solution in the ball* B^p_ρ *of the problem*

$$(3.22) \qquad \underset{\sim}{\mathcal{B}}(\underset{\sim}{U}^n) = \frac{n}{N}\underset{\sim}{f}, \quad 0 \le n \le N .$$

Proof : The proof of the well-defined character and of the convergence of Euler's method are quite classical (cf. e.g. CROUZEIX-MIGNOT [1983]) and for this reason shall not be reproduced here (suffice it to say that they essentially rely on assumptions (3.15) and (3.16)). It simply remains to show that the sequence $\underset{\sim}{u}^n$ defined in (3.17)-(3.19) does indeed coincide with the sequence found in the description of the incremental method.

Expanding the difference $\underset{\sim}{\mathcal{B}}(\underset{\sim}{u} + \underset{\sim}{v}) - \underset{\sim}{\mathcal{B}}(\underset{\sim}{u})$ for arbitrary elements $\underset{\sim}{u}, \underset{\sim}{v} \in \underset{\sim}{V}^p(\Omega)$, one finds, using (3.2) :

$$(3.23) \qquad \mathcal{B}_i'(\underset{\sim}{u})\underset{\sim}{v} = -\partial_j(a_{ijpq}\partial_p v_q + a_{ijpq}\partial_p u_m \partial_q v_m + a_{kjpq}\partial_k u_i \partial_p v_q + a_{kjpq}\partial_k u_i \partial_p u_m \partial_q v_m + a_{kjpq} e_{pq}(\underset{\sim}{u})\partial_k v_i).$$

In view of (3.18), one iteration of Euler's method can also be written as

$$(3.24) \qquad \mathcal{B}_i'(\underset{\sim}{u}^n)\delta\underset{\sim}{u}^n = \frac{1}{N} f_i \quad , \quad \delta\underset{\sim}{u}^n = \underset{\sim}{u}^{n+1} - \underset{\sim}{u}^n .$$

But, using (3.23), eqs. (3.24) are seen to coincide with eqs. (2.17).

□

Final Remarks : In BERNADOU, CIARLET and HU [1982a, 1982b], we shall extend the present analysis to *more general constitutive equations* (of the form (1.9)) and we shall also consider the effect of approximating the solution of the linearized problems (2.17)-(2.18) by *finite element methods* (as is always the case in actuel computations).

□

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